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On the excitation of long nonlinear water waves by a moving pressure distribution

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A study is made of the wave disturbance generated by a localized steady pressure distribution travelling at a speed close to the long-water-wave phase speed on water of finite depth. The linearized equations of motion are first used to obtain the large-time asymptotic behaviour of the disturbance in the far field; the linear response consists of long waves with temporally growing amplitude, so that the linear approximation eventually breaks down owing to finite-amplitude effects. A nonlinear theory is developed which shows that the generated waves are actually of bounded amplitude, and are governed by a forced Korteweg–de Vries equation subject to appropriate asymptotic initial conditions. A numerical study of the forced Korteweg–de Vries equation reveals that a series of solitons are generated in front of the pressure distribution.

1. Introduction

The generation of water waves by a two-dimensional pressure distribution, which is applied at the free surface and is travelling at a constant speed, is discussed by Stoker (1957), using the linearized water-wave theory. The large-time asymptotic behaviour of the generated wave disturbance in the far field is extracted from the exact solution of an initial-value problem. Alternatively, the characteristics of the excited waves, when the initial transients have died out, can be obtained directly by simple kinematic arguments (Whitham 1974): the main disturbance consists of a uniform sinusoidal wavetrain with phase speed equal to the propagation speed U of the pressure excitation; the position of the waves relative to the source depends on the magnitude of their group velocity. In particular, for waves on water of uniform depth, neglecting surface-tension effects, the phase and group velocities are increasing functions of the wavelength, the group velocity being smaller than the phase velocity. Accordingly, waves are excited in the far field only if the pressure disturbance travels at a speed slower than the long-water-wave speed, c_0 ($U < c_0$), and appear behind the source. In the case $U > c_0$, only transients are generated, which decay in the far field.

Stoker (1957) noted, however, that, when the pressure source travels at the long-water-wave speed ($U = c_0$), the linear response becomes unbounded. The unbounded growth predicted by the linear theory as U approaches c_0 can be easily understood; for the energy transferred by the travelling pressure distribution to the water cannot be radiated away from the source owing to the fact that the group velocity of the generated waves approaches U. Thus the linear theory cannot be uniformly valid: however small the pressure excitation, the generated disturbance eventually attains a finite amplitude and the linearized equations of motion are invalidated; the neglected nonlinear terms become of crucial importance in the evolution of the response in the finite-amplitude regime.

More recently, Wu & Wu (1982) examined the nonlinear behaviour of the generated wave disturbance close to critical conditions ($U = c_0$). Using the Boussinesq long-wave approximation, they investigated numerically the waves generated by a certain travelling pressure distribution. Their results indicate the appearance of solitons in front of the pressure source.

In the present study, which aims at describing the excited wave disturbance from the early stage, when the linearized equations of motion are valid, to the nonlinear regime, a different approach is adopted: first the results of the linear theory at the criticial speed $U = c_0$ are reexamined; a uniformly valid expression for the dominant disturbance in the far field is derived, which shows that the linear response becomes unbounded for large times only, and not for large distances from the pressure excitation (at a fixed time) as claimed by Stoker (1957). Secondly the evolution of the generated waves in the finite-amplitude regime is investigated by asymptotically matching the nonlinear response to the linear far-field response. It is found that the nonlinear response is of finite, but relatively large, amplitude and is governed by a forced Korteweg–de Vries equation subject to appropriate asymptotic initial conditions. A numerical study of the forced Korteweg–de Vries equation reveals that a series of solitons are generated in front of the pressure distribution, in qualitative agreement with the results of Wu & Wu (1982). Finally some comments are made regarding the agreement of the theoretical predictions with the solitons observed by Huang *et al.* (1982) in their experiments with ships moving in very shallow water.

Debnath & Rosenblat (1969) discuss the more general problem of surface-wave excitation by a time-harmonic travelling pressure distribution. They find critical speeds at which the linear theory predicts an unbounded response. It appears that the methods developed here might also be extended to these cases.

2. Formulation and linear solution

Consider water of uniform depth h, $0 < y < h$, under the action of a pressure distribution $p(x)$, applied at the free surface and travelling at a constant speed. A frame of reference is adopted such that the pressure is stationary and a uniform current U exists in the water. The classical gravity water-wave problem is formulated in terms of the velocity potential $\Phi = Ux - \frac{1}{2}U^2t + \phi(x, y, t)$ and the free-surface elevation $y = h + \eta(x, t)$. It proves convenient to use dimensionless (primed) variables:

$$x = hx', \quad y = hy', \quad t = \frac{h}{c_0}t', \quad \eta = a\eta', \quad \phi = \frac{agh}{c_0}\phi', \quad p = ag\rho p',$$

where g is the gravitational acceleration, ρ is the water density, $c_0 = (gh)^{\frac{1}{2}}$ is the long-water-wave speed and a denotes a typical perturbation amplitude. When the primes are dropped, the problem consists of Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad (0 < y < 1 + \epsilon\eta), \tag{1}$$

subject to the kinematic and dynamic conditions at the free surface, and the bottom boundary condition

$$\eta_t + \frac{U}{c_0}\eta_x + \epsilon\phi_x\eta_x = \phi_y \quad (y = 1 + \epsilon\eta), \tag{2}$$

$$\phi_t + \eta + \frac{U}{c_0}\phi_x + \tfrac{1}{2}\epsilon(\phi_x^2 + \phi_y^2) = -p(x) \quad (y = 1 + \epsilon\eta), \tag{3}$$

$$\phi_y = 0 \quad (y = 0), \tag{4}$$

where the dimensionless parameter $\epsilon = a/h$ is a measure of nonlinearity. In addition, initial conditions are needed, and, following Stoker (1957), it is supposed that the pressure distribution is switched on at $t = 0$, the water being undisturbed for $t \leqslant 0$, so that

$$\eta = \eta_t = \phi = \phi_t = 0 \quad (t = 0). \tag{5}$$

The linear response is the solution of the linear initial-value problem (1)–(5) ($\epsilon = 0$)† and is readily found by taking Fourier transforms in x:

$$\phi(x, y, t) = \int_{-\infty}^{\infty} \hat{\phi}(k; y, t)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k. \tag{6}$$

Elementary manipulations then show that (see Stoker 1957)

$$\hat{\phi}(k; y, t) = A(k; t) \cosh ky, \tag{7}$$

where

$$A(k; t) = \frac{\mathrm{i}kU\hat{p}}{c_0 \cosh k}\left\{\frac{1}{f_+ f_-} + \frac{1}{2(k \tanh k)^{\frac{1}{2}}}\left(\frac{\mathrm{e}^{-\mathrm{i}tf_+}}{f_+} - \frac{\mathrm{e}^{-\mathrm{i}tf_-}}{f_-}\right)\right\}, \tag{8}$$

with

$$f_{\pm}(k) = \frac{U}{c_0} k \pm (k \tanh k)^{\frac{1}{2}}, \tag{9}$$

$\hat{p}(k)$ being the transform of $p(x)$. Having determined the perturbation velocity potential ϕ, the free-surface elevation can be obtained from the linearized form of (3).

It can be easily verified that the integral in (6) is non-singular, although the individual terms in (8) can have poles on the real axis. It proves convenient for the subsequent discussion, however, to work with the velocity component ϕ_x and separate it to steady and time-varying parts:

$$\phi_x(x, y, t) = \phi_x^{\mathrm{s}}(x, y) + \phi_x^{\mathrm{t}}(x, y, t), \tag{10}$$

where

$$\phi_x^{\mathrm{s}} = -\frac{U}{c_0}\int_C \frac{k^2 \hat{p} \cosh ky}{f_+ f_- \cosh k}\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k, \tag{11}$$

$$\phi_x^{\mathrm{t}} = \frac{U}{2c_0}\int_C \frac{k^2 \hat{p} \cosh ky\, \mathrm{e}^{\mathrm{i}kx}}{(k \tanh k)^{\frac{1}{2}} \cosh k}\left(\frac{\mathrm{e}^{-\mathrm{i}tf_-}}{f_-} - \frac{\mathrm{e}^{-\mathrm{i}tf_+}}{f_+}\right) \mathrm{d}k; \tag{12}$$

the contour C extends from $-\infty$ to ∞ and is indented with semicircles to pass below any poles on the real axis.

As shown by Stoker (1957), for $U \neq c_0$, the response eventually consists of the steady part ϕ_x^{s} alone: if $U < c_0$ a uniform wave is found behind the source and no disturbance ahead, while if $U > c_0$ no disturbance at all exists far from the pressure; the unsteady part ϕ_x^{t} represents transients which die out at any position as $t \rightarrow \infty$. However, at critical conditions ($U = c_0$) these conclusions are not valid. In fact, as will be seen in §3, the dominant disturbance in the far field is provided by ϕ_x^{t}.

3. Asymptotic behaviour in the far field

As already indicated, the case of primary interest here is when $U/c_0 = 1$, for, in this instance, the linear theory predicts an unbounded response for large times. Accordingly, the asymptotic behaviour of ϕ_x^{s} and ϕ_x^{t} is sought under the condition $U/c_0 = 1$.

† Since no steady state exists under the condition $U/c_0 = 1$, which is of interest here, the simpler approach discussed by Whitham (1974), which avoids the solution of an initial-value problem, gives a singular result.

First consider the steady term ϕ_x^{s}. From (9) it is clear that the integrand in (11) has a double pole at $k = 0$. Thus, using the residue theorem, we write

$$\phi_x^{\mathrm{s}} = 6\pi H(x)\,(\hat{p}(0)\,x - \mathrm{i}\hat{p}'(0)) + \phi_x^{\mathrm{s}\pm} \quad (x \gtrless 0), \tag{13}$$

where $H(x)$ is the Heaviside unit-step function, and

$$\phi_x^{\mathrm{s}\pm} = -\int_{C^\pm} \frac{k^2\hat{p}\cosh ky}{f_+f_-\cosh k}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k; \tag{14}$$

the contour C^+ is now indented with a semicircle to pass above the pole at $k = 0$, and, for convenience, the notation C^- is used for the original contour C.

It should be noted that $\phi_x^{\mathrm{s}\pm} \to 0$ as $x \to \pm\infty$; both integrals go to zero at least as $1/x$ since no points of stationary phase exist and the contributions of the semicircles of $C^\pm$ vanish as $x \to \pm\infty$. Therefore, far from the pressure distribution, the steady-state term ϕ_x^{s} gives a disturbance growing like x for $x > 0$, and makes no contribution to $x < 0$.

Consider next the time-dependent part ϕ_x^{t}; it can be rewritten as

$$\phi_x^{\mathrm{t}} = \frac{1}{2}\int_{C^-} \frac{k^2\hat{p}\cosh ky\,\mathrm{e}^{\mathrm{i}kx}}{(k\tanh k)^{\frac{1}{2}}\cosh k}\,\frac{\mathrm{e}^{-\mathrm{i}tf_-}}{f_-}\,\mathrm{d}k - \frac{1}{2}\int_{-\infty}^{\infty} \frac{k^2\hat{p}\cosh ky\,\mathrm{e}^{\mathrm{i}kx}}{(k\tanh k)^{\frac{1}{2}}\cosh k}\,\frac{\mathrm{e}^{-\mathrm{i}tf_+}}{f_+}\,\mathrm{d}k; \tag{15}$$

the path of integration in the second integral in (15) has been deformed back to the real axis, since the integrand is regular at $k = 0$.

We are interested in the asymptotic behaviour of ϕ_x^{t} in the far field: $t \to \infty$, $m = x/t$ fixed; the main contribution comes from the neighbourhood of points of stationary phase k_0:

$$f'_\pm(k_0) = m. \tag{16}$$

It is clear from (9) that the second integral in (15) has a second-order stationary point $(f''_+(k_0) = 0, f'''_+(k_0) \neq 0)$ at $k_0 = 0$ for $m = 2$, and the dominant contribution decays like $t^{-\frac{1}{3}}$ by the standard stationary-phase argument. However, as will be shown, this contribution is negligible compared with that of the first integral in (15). Accordingly, attention is focused on the first integral I, which is re-expressed by the residue theorem:

$$\phi_x^{\mathrm{t}} \sim I(x,t) = -6\pi H(x)\,(\hat{p}(0)\,x - \mathrm{i}\hat{p}'(0)) + I^\pm(x,t) \quad (x \gtrless 0), \tag{17}$$

where

$$I^\pm(x,t) = \frac{1}{2}\int_{C^\pm} \frac{k^2\hat{p}\cosh ky\,\mathrm{e}^{\mathrm{i}kx}}{(k\tanh k)^{\frac{1}{2}}\cosh k}\,\frac{\mathrm{e}^{-\mathrm{i}tf_-}}{f_-}\,\mathrm{d}k, \tag{18}$$

and $C^\pm$ are defined as in (14). It is important to notice that the residue contribution to $I(x,t)$ in (17) cancels exactly the residue contribution to ϕ_x^{s} in (13).

It remains to examine the far-field asymptotic behaviour of $I^\pm(x,t)$. The main contribution to the integrals in (18) comes from the neighbourhood of $k = 0$ for $m \to 0$: $k = 0$ is a second-order point of stationary phase and a double pole of the integrand. Thus expanding about $k = 0$

$$f_-(k) = \tfrac{1}{6}k^3 + \dots, \tag{19a}$$

$$\frac{k^2\hat{p}\cosh ky}{(k\tanh k)^{\frac{1}{2}}\cosh k f_-} = \frac{6\hat{p}(0)}{k^2} + \dots, \tag{19b}$$

and with the change of variables $s = (\frac{1}{2}t)^{\frac{1}{3}}k$, the uniformly valid expressions

$$I^\pm(x,t) \sim 3\hat{p}(0)\left(\frac{t}{2}\right)^{\frac{1}{3}} \int_{C^\pm} \frac{\exp[\mathrm{i}(\xi s - \frac{1}{3}s^3)]}{s^2}\,\mathrm{d}s \quad (\xi \gtrless 0) \tag{20}$$

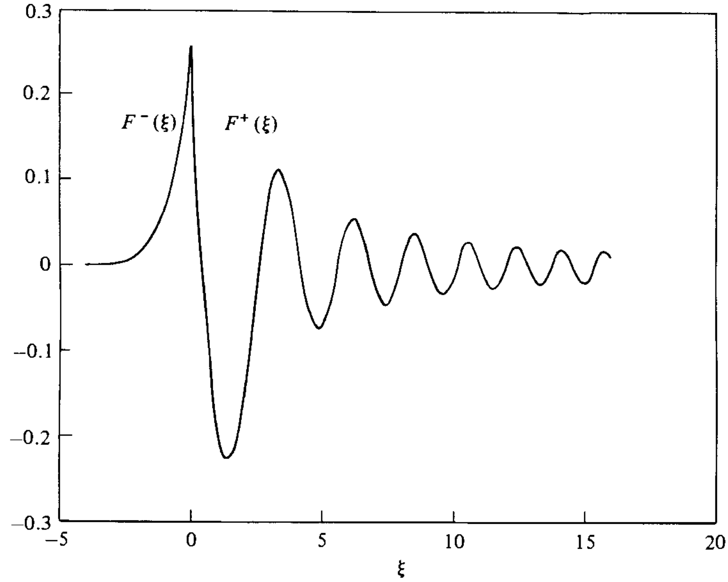

FIGURE 1. The functions $F^+(\xi)$, $\xi > 0$, and $F^-(\xi)$, $\xi < 0$.

are obtained, where $\xi = x(2/t)^{\frac{1}{3}}$. The integrals in (20) can be evaluated asymptotically by the method of steepest descent to show that $I^{\pm}(x,t)$ tend to zero as $\xi \rightarrow \pm\infty$:

$$I^+(x,t) \sim \frac{6\pi^{\frac{1}{2}}\hat{p}(0)}{\xi^{\frac{5}{4}}}\left(\frac{t}{2}\right)^{\frac{1}{3}} \sin\left(\tfrac{2}{3}\xi^{\frac{3}{2}}+\tfrac{1}{4}\pi\right) \quad (\xi \rightarrow +\infty), \tag{21}$$

$$I^-(x,t) \sim \frac{-3\pi^{\frac{1}{2}}\hat{p}(0)}{|\xi|^{\frac{5}{4}}}\left(\frac{t}{2}\right)^{\frac{1}{3}} \exp\left(-\tfrac{2}{3}|\xi|^{\frac{3}{2}}\right) \quad (\xi \rightarrow -\infty). \tag{22}$$

Alternative expressions for $I^{\pm}(x,t)$ can be obtained from (20) by noting that

$$\begin{aligned} I^{\pm}_{\xi\xi} &\sim -3\hat{p}(0)\left(\tfrac{1}{2}t\right)^{\frac{1}{3}} \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\left(\xi s-\tfrac{1}{3}s^3\right)\right] \mathrm{d}s \\ &= -6\pi\hat{p}(0)\left(\tfrac{1}{2}t\right)^{\frac{1}{3}} \mathrm{Ai}\,(-\xi), \end{aligned} \tag{23}$$

Ai being the standard Airy function; integrating (23) twice, using (21) and (22), one finds

$$I^{\pm}(x,t) \sim -6\pi\hat{p}(0)\left(\tfrac{1}{2}t\right)^{\frac{1}{3}} F^{\pm}(\xi) \quad (\xi \gtrless 0), \tag{24}$$

where

$$F^+(\xi) = \int_{-\infty}^{-\xi} \mathrm{d}s \int_{-\infty}^{s} \mathrm{Ai}\,(\sigma)\,\mathrm{d}\sigma \quad (\xi > 0), \tag{25}$$

$$F^-(\xi) = \int_{-\xi}^{\infty} \mathrm{d}s \int_{s}^{\infty} \mathrm{Ai}\,(\sigma)\,\mathrm{d}\sigma \quad (\xi < 0). \tag{26}$$

Finally, combining (13), (17) and (24), the asymptotic behaviour of ϕ_x as $t \rightarrow \infty$ reads

$$\phi_x \sim \phi_x^{\mathrm{s}\pm} - 6\pi\hat{p}(0)\left(\tfrac{1}{2}t\right)^{\frac{1}{3}} F^{\pm}(\xi) \quad (x \gtrless 0). \tag{27}$$

Therefore, at a fixed position, the dominant disturbance grows like $t^{\frac{1}{3}}$ as $t \rightarrow \infty$ (assuming $\hat{p}(0) \neq 0$):

$$\phi_x \sim 6\pi\hat{p}(0)\left(\tfrac{1}{2}t\right)^{\frac{1}{3}} \mathrm{Ai}'(0), \tag{28}$$

where the fact $F^+(0) = F^-(0) = -\mathrm{Ai}'(0) = 0.25882$ has been used. On the other hand, for large distances ($|\xi| \rightarrow \infty$, $t \rightarrow \infty$), it is clear from (21) and (22) (see figure 1) that the disturbance decays exponentially for $x < 0$ and algebraically for $x > 0$.

The conclusions reached above are at variance with those of Stoker (1957), who finds that the disturbance becomes unbounded in time like $t^{\frac{1}{3}}$ and in space like x. The cause of the disagreement can be traced back to the asymptotic evaluation of ϕ_x^{t}: if the expression (15) for ϕ_x^{t} is differentiated with respect to t, Watson's lemma can be used directly to find that ϕ_{xt}^{t} (the integrand of which is now regular at $k = 0$) behaves like $t^{-\frac{2}{3}}$; on these grounds, Stoker (1957) inferred that ϕ_x^{t} grows like $t^{\frac{1}{3}}$ and, using (13), concluded that ϕ_x grows in space like x as well. However, it is clear now that, by examining ϕ_{xt}^{t} alone, the residue contribution to ϕ_x^{t} in (17) was missed, and thus the term proportional to x in (13) did not cancel.

4. Nonlinear analysis

As shown in §3, the linear theory predicts an unbounded response as $t \to \infty$. Thus the originally negligible nonlinear terms eventually become important and the unbounded growth is modified.† An asymptotic theory which accounts for the finite-amplitude effects ($\epsilon \ll 1$) is developed in this section.

The timescale on which the nonlinear effects come into play can be found by referring back to (27): for $t \gg 1$ the amplitude of the disturbance grows like $\epsilon t^{\frac{1}{3}}$, while the dispersive effects, which are proportional to the square of the wavenumber, decay like $t^{-\frac{2}{3}}$; a balance is reached when $t = O(\epsilon^{-1})$. Accordingly, the slow time $T = \epsilon t$ and the slow spatial variable $X = \epsilon^{\frac{1}{3}} x$ (suggested by the definition of ξ) are defined. Furthermore, when $t = O(\epsilon^{-1})$, $\phi = O(\epsilon^{-\frac{2}{3}})$ and $\eta = O(\epsilon^{-\frac{1}{3}})$, so that new rescaled variables, appropriate in the far field, are adopted:

$$\phi = \epsilon^{-\frac{2}{3}} \tilde{\phi}(X, T; y), \quad \eta = \epsilon^{-\frac{1}{3}} \tilde{\eta}(X, T) \tag{29}$$

where now $\tilde{\phi}$ and $\tilde{\eta}$ are assumed to be $O(1)$.

In terms of the new variables, the governing equations (1)–(4) read:

$$\epsilon^{\frac{2}{3}} \tilde{\phi}_{XX} + \tilde{\phi}_{yy} = 0 \quad (0 < y < 1 + \epsilon^{\frac{2}{3}} \tilde{\eta}), \tag{30}$$

$$\epsilon^{\frac{2}{3}} \tilde{\eta}_X + \epsilon^{\frac{4}{3}} (\tilde{\eta}_T + \lambda \tilde{\eta}_X + \tilde{\phi}_X \tilde{\eta}_X) = \tilde{\phi}_y \quad (y = 1 + \epsilon^{\frac{2}{3}} \tilde{\eta}), \tag{31}$$

$$\tilde{\eta} + \tilde{\phi}_X + \tfrac{1}{2} \tilde{\phi}_y^2 + \epsilon^{\frac{2}{3}} (\tilde{\phi}_T + \lambda \tilde{\phi}_X + \tilde{\phi}_X^2) = -\epsilon^{\frac{1}{3}} p(X/\epsilon^{\frac{1}{3}}) \quad (y = 1 + \epsilon^{\frac{2}{3}} \tilde{\eta}), \tag{32}$$

$$\tilde{\phi}_y = 0 \quad (y = 0), \tag{33}$$

and the possibility of being slightly off critical conditions, $U/c_0 = 1 + \lambda \epsilon^{\frac{2}{3}}$, $\lambda = O(1)$, has been included.

An asymptotic approximation to (30)–(33) is sought in the far field ($X = O(1)$, $T = O(1)$, $\epsilon \to 0$). The procedure to be followed in some respects parallels the derivation of the nonlinear long-wave approximation to the full water-wave theory (Whitham 1974): the velocity potential $\tilde{\phi}$ is expressed as an expansion in powers of $\epsilon^{\frac{1}{3}} y$, which satisfies Laplace's equation (30) and the bottom boundary condition (33):

$$\tilde{\phi}(X, T; y) = f(X, T) - \frac{\epsilon^{\frac{2}{3}} y^2}{2!} f_{XX} + \frac{\epsilon^{\frac{4}{3}} y^4}{4!} f_{XXXX} + \dots, \tag{34}$$

and, upon substitution in the free-surface conditions (31) and (32), two equations for f and $\tilde{\eta}$ are obtained:

$$\tilde{\eta}_X + f_{XX} + \epsilon^{\frac{2}{3}} (\tilde{\eta}_T + \lambda \tilde{\eta}_X + \tilde{\eta}_X f_X + \tilde{\eta} f_{XX} - \tfrac{1}{6} f_{XXXX}) = O(\epsilon^{\frac{4}{3}}), \tag{35}$$

$$\tilde{\eta} + f_X + \epsilon^{\frac{2}{3}} (f_T + \lambda f_X + f_X^2 - \tfrac{1}{2} f_{XXX}) = -\epsilon^{\frac{1}{3}} p(X/\epsilon^{\frac{1}{3}}) + O(\epsilon^{\frac{4}{3}}). \tag{36}$$

† The effects of viscosity, which are not included here, are expected to be negligible since we are dealing with long waves.

However, $$\epsilon^{-\frac{1}{3}}p(X/\epsilon^{\frac{1}{3}}) \to 2\pi\hat{p}(0)\,\delta(X) \quad (\epsilon \to 0), \tag{37}$$

where $\delta(X)$ is the Dirac delta function, so that, differentiating (36) with respect to X, using (35), and $f_X = -\tilde{\eta}+O(\epsilon^{\frac{2}{3}})$, a single equation for $\tilde{\eta}$ is obtained to leading order:

$$\tilde{\eta}_T+\lambda\tilde{\eta}_X-2\tilde{\eta}\tilde{\eta}_X-\tfrac{1}{6}\tilde{\eta}_{XXX} = \pi\hat{p}(0)\,\delta'(X). \tag{38}$$

Not unexpectedly, the homogeneous part of (38) is the Korteweg–de Vries (KdV) equation. Normally, the nonlinear long-wave approximation to the full water-wave theory leads to the Boussinesq equations; the KdV equation is derived therefrom under the further assumption of waves propagating in one direction only. However, in the present problem, the KdV equation is obtained directly, in accordance with the fact that the dominant disturbance follows the moving pressure distribution. (The part of the disturbance moving in the opposite direction is given, according to the linear theory, by the second integral in (15), and, as already indicated, is negligible.)

The appropriate initial conditions for the solution of (38) are obtained by matching asymptotically the finite-amplitude response to the result predicted by the linear theory: at intermediate times $t \gg 1$ with $T \ll 1$, the linear theory is valid; thus, recalling (27) and the fact $\eta \sim -\phi_x$, it follows that

$$\tilde{\eta}(X,T) \sim 6\pi\hat{p}(0)\,(\tfrac{1}{2}T)^{\frac{1}{3}}F^{\pm}(\xi) \quad (T \ll 1). \tag{39}$$

It can be readily verified that (39) satisfies the linearized version of (38) (with $\lambda = 0$) for $\xi \gtrless 0$: setting $\tilde{\eta}(X,T) = T^{\frac{1}{3}}g(\xi)$, direct substitution in (38) yields

$$g_{\xi\xi\xi}+\xi g_{\xi}-g = 0 \quad (\xi \neq 0), \tag{40}$$

or $$g_{\xi\xi\xi\xi}+\xi g_{\xi\xi} = 0 \quad (\xi \neq 0); \tag{41}$$

the relevant solution is proportional to $\mathrm{Ai}\,(-\xi)$, in agreement with (23). Furthermore, since the functions $F^{\pm}(\xi)$ defined in (25) and (26) satisfy the conditions†

$$F^{+}(0) = F^{-}(0) = -\mathrm{Ai}'\,(0) = 0.25882, \quad F^{+}_{\xi}(0) = -\tfrac{2}{3},$$
$$F^{-}_{\xi}(0) = \tfrac{1}{3}, \quad F^{+}_{\xi\xi}(0) = F^{-}_{\xi\xi}(0) = \mathrm{Ai}\,(0) = 0.35503,$$

it is clear that the appropriate jump conditions at $X = 0$, imposed by the right-hand side of (38), are met.

If $\lambda \neq 0$, the appropriate solution of the linearized forced KdV equation takes the form

$$\tilde{\eta}(X,T) = -3\hat{p}(0)\int_{C^{\pm}}\frac{\exp\left[\mathrm{i}s(X-\lambda T)-\frac{1}{6}\mathrm{i}s^3T\right]}{s^2+6\lambda}\,\mathrm{d}s \quad (X \gtrless 0), \tag{42}$$

where C^{+} passes above the poles of the integrand, while C^{-} passes below. Using the method of steepest descent, it can be shown that for $\lambda > 0$ the disturbance eventually decays as $T \to \infty$ for any fixed X; for $\lambda < 0$ the disturbance decays if $X < 0$, but forms a steady wave for $X > 0$:

$$\tilde{\eta}(X,T) \sim 6\pi\hat{p}(0)\frac{\sin\,(-6\lambda)^{\frac{1}{2}}X}{(-6\lambda)^{\frac{1}{2}}}. \tag{43}$$

These conclusions are consistent with the predictions of the full linear theory.

According to (38) and (39), the evolution of the generated wave disturbance in the far field depends on $\hat{p}(0)$ only, which is proportional to the total force acting on the water surface and is assumed to be finite; the precise details of the pressure

† Equation (40), together with the above conditions at $\xi = 0$, can be used to compute numerically the functions $F^{\pm}(\xi)$; the results shown in figure 1 were computed in this way, using a standard fourth-order Runge–Kutta scheme.

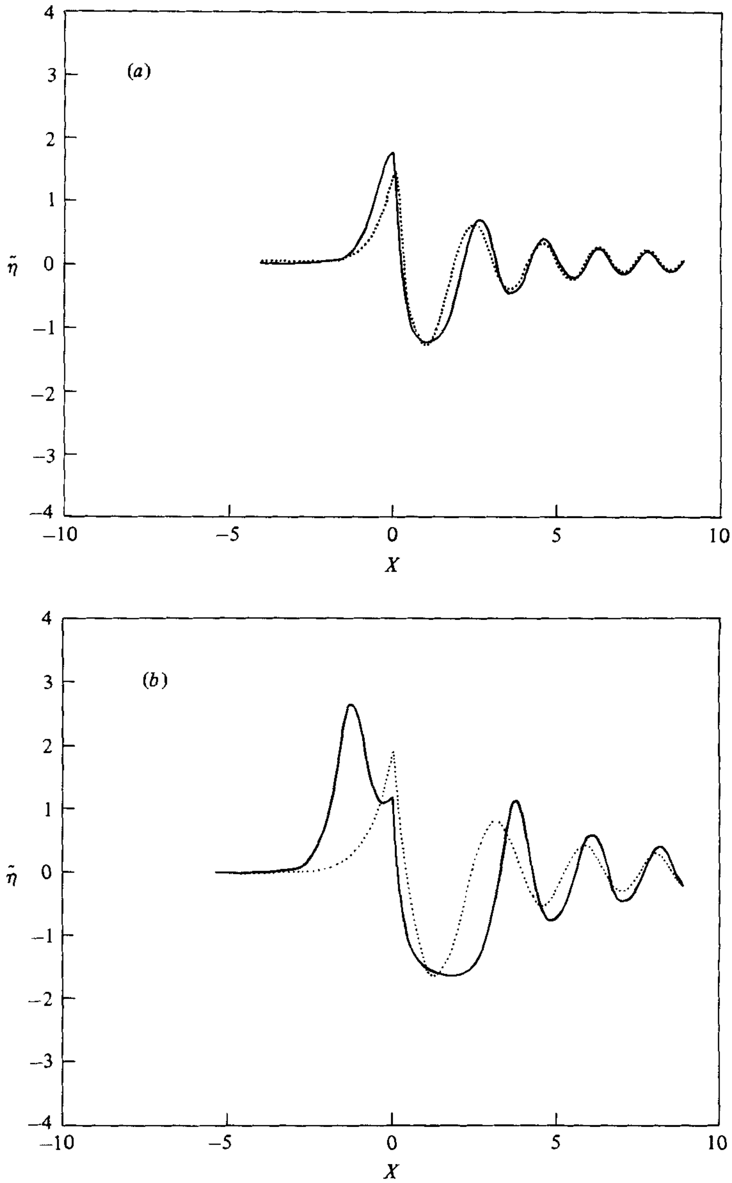

FIGURE 2(a, b). For caption see facing page.

distribution are of significance only in a small region around $X = 0$, where the initial condition (39) exhibits a discontinuous slope (figure 1). In reality, of course, the disturbance is smooth at $X = 0$ owing to the contribution $\phi_x^{s\pm}$ in (27); such terms can be consistently neglected in (39) since they are small for $|X| \gg \epsilon^{\frac{1}{3}}$, where (38) applies. (Strictly speaking, (38) is not valid in an $O(\epsilon^{\frac{1}{3}})$ region around $X = 0$, where the disturbance varies rapidly (with respect to X) and is governed by the full water-wave theory. However, to the leading-order approximation used here, the width of this region has shrunk to zero.)

The scalings (29) imply that the far-field disturbance is of relatively large amplitude ($O(\epsilon^{\frac{2}{3}})$ for an $O(\epsilon)$ pressure excitation), but remains bounded; the unbounded response

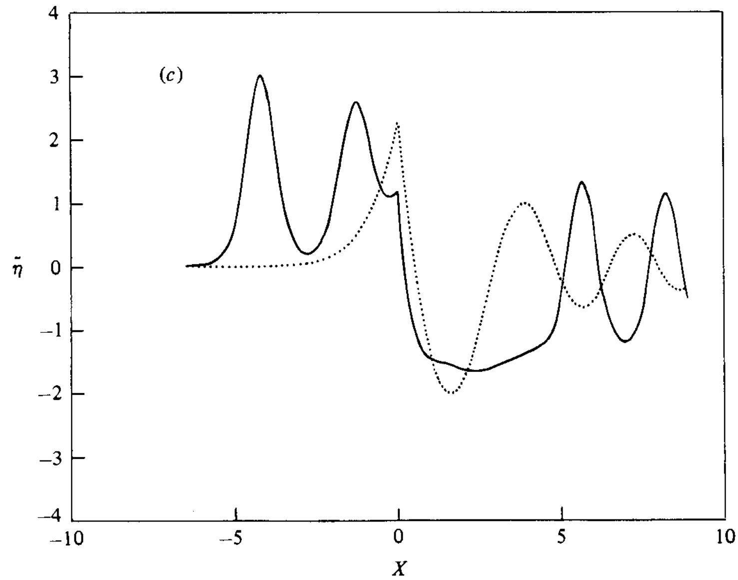

FIGURE 2. The evolution of the wave disturbance at critical conditions ($\lambda = 0$): ——, nonlinear response; ..., linear response. (*a*) $T = 0.8$; (*b*) 1.7; (*c*) 3.2.

predicted by the linear theory is just the first term in the expansion of the nonlinear response for small T. The evolution of the disturbance for $T = O(1)$ is governed by the inhomogeneous KdV equation (38) with the initial condition (39); the appropriate solution is unknown, however. Accordingly, the question whether the nonlinear response evolves to solitons, or disperses out, or even gives a cnoidal wave (since for $\lambda < 0$ the linear response consists of a periodic sinusoidal wave in $X > 0$ as shown in (43)) still remains. It is answered in §5 by a numerical investigation.

5. Numerical results

The inhomogeneous KdV equation (38) was solved numerically by an explicit finite-difference scheme, suggested by Vliegenthart (1971). With the notation

$$\tilde{\eta}_j^n = \tilde{\eta}(j\,\Delta X,\, n\,\Delta T),$$

$\tilde{\eta}_j^{n+1}$ is approximated by a Taylor-series expansion for $X \neq 0$:

$$\tilde{\eta}_j^{n+1} = \tilde{\eta}_j^n + \Delta T\, \tilde{\eta}_{Tj}^n + \tfrac{1}{2}\Delta T^2\, \tilde{\eta}_{TTj}^n, \tag{44}$$

with $O(\Delta T^3)$ local truncation error. Using (38), $\tilde{\eta}_T$ and $\tilde{\eta}_{TT}$ can be expressed in terms of $\tilde{\eta}$, $\tilde{\eta}_X$, $\tilde{\eta}_{XX}$, $\tilde{\eta}_{XXX}$, $\tilde{\eta}_{XXXX}$ and $\tilde{\eta}_{XXXXXX}$, which are then approximated at $(j\,\Delta X,\, n\,\Delta T)$ by second-order centred finite differences. At $X = 0$ the jump conditions implied by the right-hand side of (38) are imposed:

$$\left.\begin{aligned} \tilde{\eta}(X = 0^-, T) = \tilde{\eta}(X = 0^+, T), \quad \tilde{\eta}_X(X = 0^-, T) - \tilde{\eta}_X(X = 0^+, T) = 6\pi\hat{p}(0), \\ \tilde{\eta}_{XX}(X = 0^-, T) = \tilde{\eta}_{XX}(X = 0^+, T) \end{aligned}\right\}, \tag{45}$$

replaced by finite differences. Furthermore, a sufficiently large domain for X was decided by numerical experiments so that, within the computational time interval, the disturbances behaved according to the linear result (42) at the boundaries of the domain. Thus, starting with the asymptotic initial condition (39) at $T = T_0 \ll 1$, the solution of (38) was advanced numerically in time.

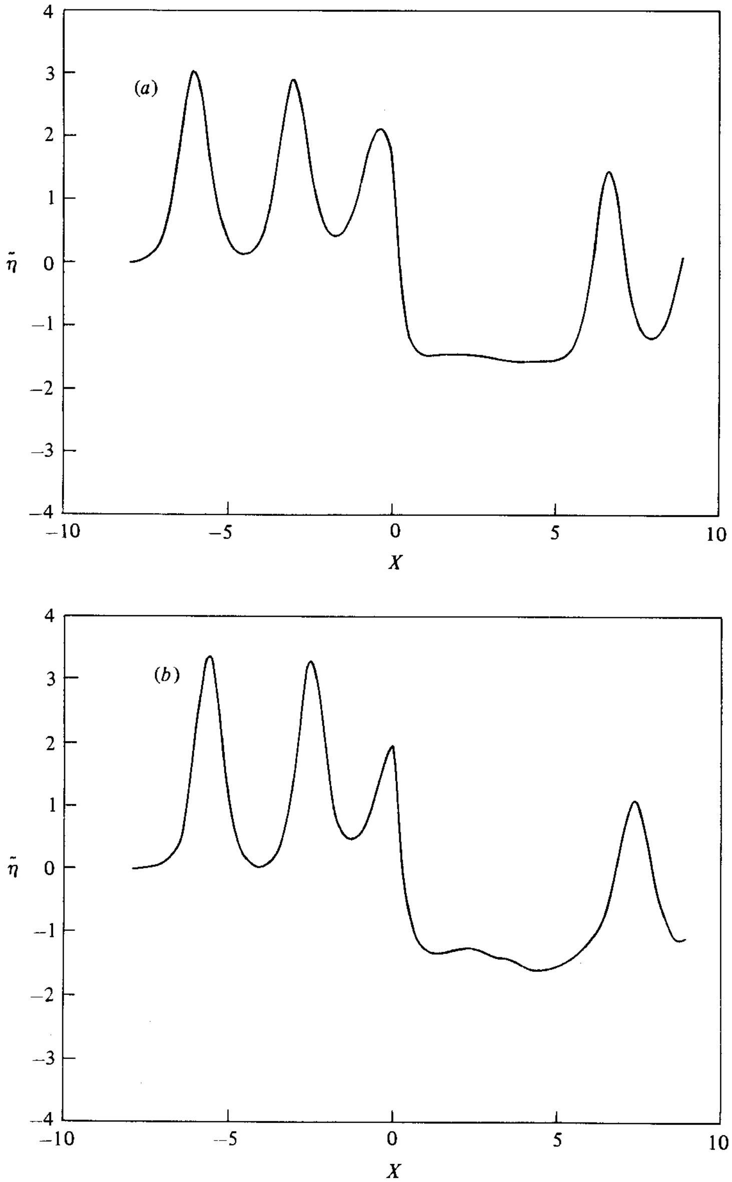

FIGURE 3(a, b). For caption see facing page.

The above numerical procedure was carried out with $\Delta X = 0.08$ and $T_0 = 0.1$. As shown by Vliegenthart (1971), the numerical scheme is only conditionally stable and is expected to be stable for $\Delta T \lesssim \Delta X^3$. (Our experience shows that, in the present problem, for $\Delta X = 0.08$, $\Delta T < 0.6 \times 10^{-3}$ is sufficient for stability.) In addition this scheme incorporates some 'numerical damping' required to resolve accurately the highly oscillatory nature of the solution for $X > 0$; preliminary numerical experiments showed that simpler finite-difference schemes, such as those used by Zabusky & Kruskal (1965) or Peregrine (1966), were not appropriate for the problem of interest.

The accuracy of the numerical calculations was checked in various ways: first, the known exact linear solution (39) was compared with the corresponding numerical solution and agreement was found to within 2 or 3 % for $T \lesssim 5$. Secondly, the

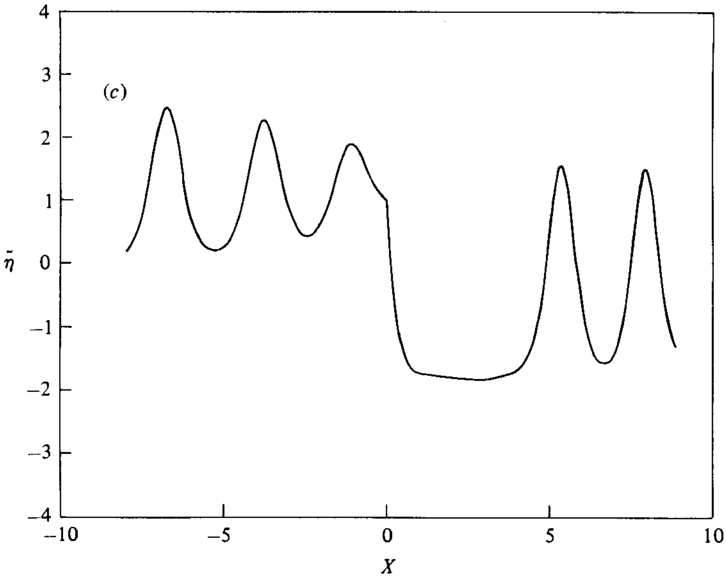

FIGURE 3. The wave disturbance at $T = 4.1$: (*a*) $\lambda = 0$; (*b*) 0.25; (*c*) -0.5.

numerical scheme was successfully tested against known soliton solutions of the KdV equation. Finally, the use of refined integration steps ΔT, ΔX, and a larger computational domain for X did not alter the results within the accuracy reported here, especially in the domain $-10 < X < 10$, where the most interesting effects occur for $T \lesssim 4.5$.

Figures 2(*a–c*) show the evolution of the wave disturbance in the nonlinear regime at critical conditions ($\lambda = 0$); a comparison with the corresponding linear solution (39) is also made. Solitons are successively generated in $X < 0$ and propagate in front of the pressure distribution: after each soliton reaches a certain equilibrium amplitude, a new soliton of slightly smaller equilibrium amplitude is released. Immediately behind the pressure excitation, a long wave trough appears (to balance more or less the amount of water used to form the solitons); at larger distances from the pressure, the wave disturbance is highly oscillatory with a larger amplitude than that predicted by the linear theory.

For supercritical or subcritical speeds ($\lambda \gtrless 0$), solitons are generated in a way qualitatively similar to the critical case described above. Figures 3(*a–c*) show the wave disturbance at $T = 4.1$ for $\lambda = 0$, 0.25 and -0.5. It is clear that both the equilibrium amplitudes of the generated solitons and the period of soliton formation increase as λ increases. However, the relative amplitude difference between successive solitons appears to decrease as λ increases.

It is interesting that the theoretical predictions concerning the generated solitons are in good agreement with the experimental results of Huang *et al.* (1982). Although the experiments involved a ship moving in a shallow-water tank of finite width, two-dimensional solitons were observed in front of the ship at speeds close to the long-water-wave speed. This points to the fact that the generation of solitons depends on the total force acting on the water-surface only, and not on the precise details of the excitation (even if the source is three-dimensional!), in accordance with the asymptotic theory. Moreover, all the trends of the theoretical predictions noted above close to critical conditions are in agreement with experiment.

The numerical calculations of Wu & Wu (1982) with the Boussinesq equations also succeed in predicting the appearance of solitons, but there are marked differences between their results and the results of this study: first, according to Wu & Wu, the generated solitons keep growing indefinitely as they propagate away from the pressure distribution; secondly, in the oscillatory tail, far behind the pressure, the wave crests are far more peaked than the troughs. The question as to whether these discrepancies are due to numerical error or some other reason still remains.

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